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MOVING AVERAGE MODELS--TIME SERIES IN M-DIMENSIONS.(U)

JAN 78 D A VOSS, C A OPRIAN, L A AROIAN

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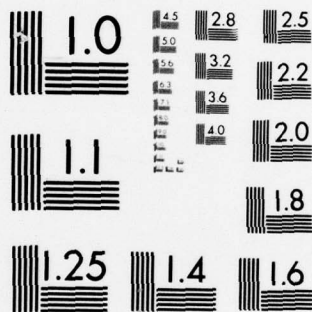
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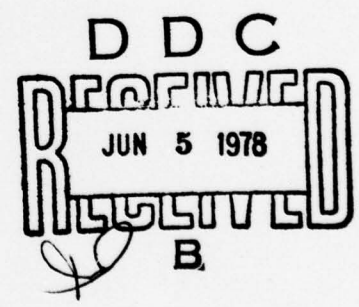
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MOVING AVERAGE MODELS--TIME SERIES  
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## MOVING AVERAGE MODELS--TIME SERIES IN M-DIMENSIONS

Page 1, Abstract, line 2, insert (see Box & Jenkins, 1976) after "well known."

Page 2, line 4,  $E(z_{x,t})$  should be  $E[z_{x,t}]$ .

Page 2, Eq. (1.2), brackets are missing in  $E[a_{x,t} a_{x+n,t-k}]$

Page 3. The following should appear as a caption below the figure:

$$\tilde{z}_{x_1, x_2, t} = \sum_{n_2=-1}^1 \sum_{n_1=-2}^2 \sum_{k=1}^r \psi_{n_1, n_2, k} a_{x_1+n_1, x_2+n_2, t-k} a_{x_1, x_2, t}$$

spatial order four in direction  $x_1$ , order two in  $x_2$ .

"Whether or..." should begin a new paragraph.

Page 3, line 2 from bottom of page "at" should read "as."

Page 4, line 1, a comma should precede  $F_{x_m}^{n_m}$  and a comma should follow  $B_{x_1}^{-n_1}$

Page 4, line 14, should read  $\sum_{i=-m}^m X S_i$ ,

Page 9, insert following Eq. (2.1.9) "Further since  $\rho_{01}\rho_{11} = (\rho_{10}\sigma_a^2)/\sigma_z^2$ ,

then we find  $\theta_1 = (\theta_2 \rho_{01})/\rho_{11}$  and in terms of  $\rho_{01}$  and  $\rho_{11}$  only,

$$\theta_2 = \frac{-\rho_{11} \pm \rho_{11} \sqrt{1 - 4(\rho_{01}^2 + \rho_{11}^2)}}{2(\rho_{01}^2 + \rho_{11}^2)} \quad (2.1.10)$$

The advantage of (2.1.9) over (2.1.10) is that  $\theta_1$  and  $\theta_2$  are unambiguously determined by  $\rho_{10}$ ,  $\rho_{11}$ , and  $\rho_{01}$ ." Also change original (2.1.10) to (2.1.11).

Page 10, following (2.2.3) insert "Note  $\rho_{-11} = \rho_{11}$ , because of the symmetry of the autocorrelation function in m dimensions."

Page 11, line 15, add after "implies" " $|\theta_1| < 1$ , again in agreement.

Finally, with  $\theta_2 = \theta_3 = 0$  we obtain the MA(2;0,0) MA(2) process of Box and Jenkins, and (2.2.5) implies..."

Page 11, line 5 from bottom, add "In (2.2.3)  $\rho_{11} = \rho_{-11}$  due to the symmetry of the autocorrelation function in m dimensions."

(over)



Page 14, following (2.4.3), add "Note  $\rho_{11} = \rho_{-11}$ ."

Page 16, following (3.1.2), add "Note  $\gamma_{110} = \gamma_{1-10}$ ."

Page 16, following (3.1.3), add "Note  $\rho_{110} = \rho_{1-10}$ ."

Page 18, line 12, after "zero" insert "Note  $\gamma_{101} = \gamma_{-101}$ ,  $\gamma_{011} = \gamma_{0-11}$ ,  
and  $\gamma_{110} = \gamma_{1-10}$ ."

Page 20, line 13, following "autocorrelations" add "Note that

$\gamma_{101} = \gamma_{-101}$ ,  $\gamma_{011} = \gamma_{0-11}$ ,  $\gamma_{110} = \gamma_{1-10}$ ,  $\gamma_{111} = \gamma_{-111}$ ,  $\gamma_{1-11} = \gamma_{-1-11}$ ."

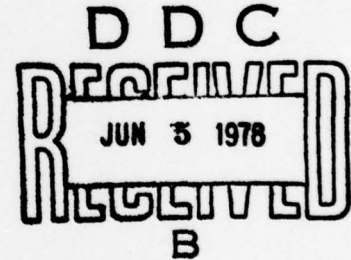
Page 21, line 17, should read "eralize to m dimensions under proper restrictions."

Page 22, add the following reference:

Taneja, V., & Aroian, L.A. (1977). Time series in m dimensions,  
autoregressive models.

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MOVING AVERAGE MODELS---TIME SERIES IN M-DIMENSIONS



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# MOVING AVERAGE MODELS--TIME SERIES IN M-DIMENSIONS

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## ABSTRACT

Stochastic models for discrete time series in the time domain are well known but such models lack consideration of spatial dependency. We expand on their work by constructing spatially dependent moving average models. Definitions of order, stationarity, invertibility, autocorrelation function, and spectrum are made as natural extensions of those in zero dimensions and are implemented in the one and two-space dimensional models.

## 1. INTRODUCTION

We describe a general linear stochastic model which supposes a time series to be generated by a linear aggregation of random shocks at various temporal and spatial locations. Letting  $x = (x_1, x_2, \dots, x_m)$ , an  $m$ -dimensional vector, the general Moving Average (MA) model of  $m$ -dimensional time series is defined by

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$$\tilde{z}_{x,t} = \sum_{n=-\infty}^{\infty} \sum_{k=1}^{\infty} \psi_{n,k} a_{x+n,t-k} + a_{x,t}, \quad (1.1)$$

where  $n = (n_1, n_2, \dots, n_m)$ ,  $\sum_{n=-\infty}^{\infty}$  denotes the repetitive sum over each component of  $n$  (i.e.,  $\sum_{n=-\infty}^{\infty} = \sum_{n_1=-\infty}^{\infty} \dots \sum_{n_2=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty}$ ), and  $\tilde{z}_{x,t} = z_{x,t} - E(z_{x,t})$  is the deviation from the mean. The white noise process  $a_{x,t}$  may be regarded as a series of independent random shocks which drive the system, so that their autocovariance function is:

$$\gamma_{n,k} = E a_{x,t} a_{x+n,t-k} = \begin{cases} \sigma_a^2, & n=0 \text{ and } k=0 \\ 0, & \text{otherwise} \end{cases} \quad (1.2)$$

and hence the autocorrelation function of white noise has the particularly simple form

$$\rho_{n,k} = \begin{cases} 1, & n=0 \text{ and } k=0 \\ 0, & \text{otherwise} \end{cases} \quad (1.3)$$

We also assume that  $\tilde{z}_{x,t}$  is a weakly stationary process, i.e.,

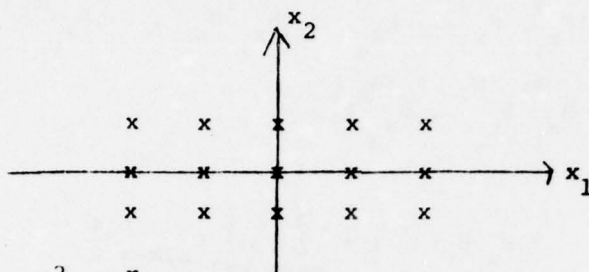
$$E[\tilde{z}_{x,t}^2] < \infty \text{ and } E[\tilde{z}_{x,t_1} \tilde{z}_{y,t_2}] = \sigma_z^2 \rho_{|x-y|, |t_1-t_2|}. \quad (1.4)$$

In this paper we explicitly focus our attention on the special case of (1.1) in which only a finite number of the coefficients are nonzero, that is:

$$\tilde{z}_{x,t} = \sum_{n=-p}^q \sum_{k=1}^r \psi_{n,k} a_{x+n,t-k} + a_{x,t} \quad (1.5)$$

If in equation (1.5) all of the coefficients,  $\psi_{n,k}$ , are nonzero, the process is called moving average of temporal order  $r$  and spatial order  $p_j + q_j$  in each space direction  $x_j$ ,  $1 \leq j \leq m$ . For example, letting  $m = 2$ ,  $p = (2,1)$ ,  $q = (2,1)$ , and  $r$  arbitrary, we have the following representative scheme:





$$\tilde{z}_{x_1, x_2, t} = \sum_{n_2=-1}^1 \sum_{n_1=-2}^2 \sum_{k=1}^r \psi_{n_1, n_2, k} a_{x_1+n_1, x_2+n_2, t-k}^{+a_{x_1, x_2, t}}$$

spatial order four in direction  $x_1$ , order two in  $x_2$ . Whether or not all the coefficients are nonzero, it is easier, from the analysis viewpoint, to represent the process (1.5) in terms of shift operators. The backward shift operator in time,  $B_t$ , is defined by

$$B_t \tilde{z}_{x, t} = \tilde{z}_{x, t-1},$$

while the backward and forward shift operators in spatial direction  $x_i$ , denoted by  $B_{x_i}$  and  $F_{x_i}$  respectively, are defined by

$$\begin{aligned} B_{x_i} \tilde{z}_{x, t} &= \tilde{z}_{x-\delta_i, t} \\ F_{x_i} \tilde{z}_{x, t} &= \tilde{z}_{x+\delta_i, t}, \end{aligned} \quad (1 < i < M)$$

where  $\delta_i = (\delta_{i1}, \delta_{i2}, \dots, \delta_{im})$  and  $\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j. \end{cases}$

Powers of these operators are defined in the usual manner, for example,

$$B_{x_i}^2 \tilde{z}_{x, t} = B_{x_i} (B_{x_i} \tilde{z}_{x, t}) = \tilde{z}_{x-2\delta_i, t}.$$

In addition, we note that the operator  $B_{x_i}$  is the inverse of  $F_{x_i}$ , that is,  $B_{x_i}^{-1} = F_{x_i}$ . In terms of these shift operators, the process (1.5) can be reformulated at

$$\tilde{z}_{x, t} = (1 + \sum_{n=-p}^q \sum_{k=1}^r \psi_{n, k} F_{x_i}^n B_t^k) a_{x, t} \quad (1.6)$$

where  $F_x^n = (F_{x_1}^{n_1}, F_{x_2}^{n_2}, \dots, F_{x_m}^{n_m}) = (B_{x_1}^{-n_1} B_{x_2}^{-n_2}, \dots, B_{x_m}^{-n_m})$

and  $F_x^n B_t^k = B_{x_1}^{-n_1} B_{x_2}^{-n_2} \dots B_{x_m}^{-n_m} B_t^k$ .

Defining

$$\Psi(B_x, B_t) \equiv 1 + \sum_{n=-p}^q \sum_{k=1}^r \psi_{n,k} F_x^n B_t^k \quad (1.7)$$

the moving average process

$$\tilde{z}_{x,t} = \Psi(B_x, B_t) a_{x,t} \quad (1.8)$$

can be thought of as the output  $\tilde{z}_{x,t}$  from a linear filter with transfer function  $\Psi(B_x, B_t)$  when the input is white noise  $a_{x,t}$ . Since the expression for  $\Psi(B_x, B_t)$  is finite, no restrictions are needed on the parameters  $\psi_{n,k}$  to ensure stationarity. The invertibility condition for the moving average process may be obtained by writing (1.8) as

$$a_{x,t} = \Psi^{-1}(B_x, B_t) \tilde{z}_{x,t}.$$

Extending the results of Box and Jenkins it can be shown that, for invertibility,  $\Pi(B_x, B_t) = \Psi^{-1}(B_x, B_t)$  must converge on  $\prod_{i=-M}^m S_i$ , where

$$\begin{aligned} S_0 &= \{B_t : |B_t| \leq 1\} \\ S_{-i} &= \{B_{x_i} : |B_{x_i}| \leq 1\} \\ S_i &= \{F_{x_i} : |F_{x_i}| \leq 1\}. \end{aligned} \quad (1 \leq i \leq m)$$

The autocovariance function of a MA process may be obtained by multiplying through (1.6) by  $\tilde{z}_{x-\ell, t-k}$ , where  $\ell = (\ell_1, \ell_2, \dots, \ell_m)$  and taking expectations. A more convenient way of obtaining the autocovariances is often via the autocovariance generating function

$$\gamma(B_x, B_t) = \sum_{\ell=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \gamma_{\ell k} B_x^\ell B_t^k, \quad (1.9)$$

in which it is noted that  $\gamma_{00}$ , the variance of the process, is the coefficient of  $B_x^0 B_t^0 \equiv 1$ , while  $\gamma_{\ell, k}$  is the coefficient of both

$B_{x,t}^{l,k}$  and of  $B_{x,t}^{-l,-k}$ ; as in Box and Jenkins, it follows that

$$\gamma(B_x, B_t) = \sigma_a^2 \psi(B_x, B_t) \psi(F_x, F_t). \quad (1.10)$$

The autocorrelation function in  $m$  dimensions is symmetrical in each variable.

It can be shown that if we substitute  $B_t = e^{-i2\pi f}$  and  $B_{x_j} = e^{-i2\pi g_j}$ , where  $i = \sqrt{-1}$  and  $1 \leq j \leq m$ , in the autocovariance generating function (1.10), we obtain the power spectrum. Thus the spectrum of the MA process is

$$\begin{aligned} p(g, f) &= 2\sigma_a^2 \psi(e^{-i2\pi g}, e^{-i2\pi f}) \psi(e^{i2\pi g}, e^{i2\pi f}) \\ &= 2\sigma_a^2 |\psi(e^{-i2\pi g}, e^{-i2\pi f})|^2, \end{aligned} \quad (1.11)$$

where  $0 < f \leq \frac{1}{2}$ ,  $0 \leq g_j \leq \frac{1}{2}$  ( $1 \leq j \leq m$ ), and  $\psi(e^{-i2\pi g}, e^{-i2\pi f}) = \psi(e^{-i2\pi g_1}, e^{-i2\pi g_2}, \dots, e^{-i2\pi g_m}, e^{-i2\pi f})$ .

In sections two and three we analyze one-dimensional and two-dimensional models, respectively. In both sections we have focused on models whose spatial and temporal orders do not exceed two. Section four contains some general remarks and indicates areas where more extensive research is needed. The following theorem generalizes a corresponding result for MA in zero dimensions.

**Theorem:** If the conditions for invertibility are satisfied, then every finite MA process in  $m$  dimensions may be expressed as an infinite AR model in  $m$  dimensions.

## 2. ONE-DIMENSIONAL TIME SERIES

In this section we consider moving average models of one-dimensional time series with temporal and spatial orders not exceeding two. For convenience and easy reference we let  $x = x_1$ , and denote by  $MA(r; p_1, q_1)$  the moving average model for one-dimensional time series of temporal order  $r$  and spatial order  $p_1 + q_1$ ; recall that  $p_1$  and  $q_1$  denote the maximum powers of the



operators  $B_x$  and  $F_x$  occurring in (1.6), respectively. This notation is an attempt to be consistent with that in Box and Jenkins, who analyzed zero-dimensional time series with moving average models of various temporal orders, that is,  $MA(r) \sim MA(r; 0, 0)$ ; this equivalence means that since the maximum powers of the spatial operators are both zero, and  $B_x^0 = F_x^0 \equiv 1$  (identity operator), we are virtually looking at the same point at different times. Formally, from (1.6) we find that

$$\tilde{z}_{x,t} = (1 + \sum_{n_1=-p_1}^{q_1} \sum_{k=1}^r \psi_{n_1,k} F_x^{n_1} B_t^k) a_{x,t}. \quad (2.1)$$

Thus with  $p_1=q_1=0$ , we obtain

$$\tilde{z}_{x,t} = a_{x,t} + \psi_{01} a_{x,t-1} + \psi_{02} a_{x,t-2} + \dots + \psi_{0r} a_{x,t-r}. \quad (2.2)$$

Deleting the first subscript in each case, as  $z_{x,t}$  depends only on  $t$ , results in the zero-dimensional moving average model in Box and Jenkins, namely:

$$\tilde{z}_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \dots + \psi_r a_{t-r}. \quad (2.3)$$

There are five general "spatial" models described by (2.1) where  $1 \leq p_1 + q_1 \leq 2$ ; we represent these diagrammatically below, where order refers to the spatial order:

|  |  |
|--|--|
| MA(r;1,0): First order backward          | $\begin{array}{c} x-1 \qquad x \\ \hline \cdot \qquad \cdot \end{array}$                         |
| MA(r;0,1): First order forward           | $\begin{array}{c} x \qquad x+1 \\ \hline \cdot \qquad \cdot \end{array}$                         |
| MA(r;1,1): Second order forward-backward | $\begin{array}{c} x-1 \qquad x \qquad x+1 \\ \hline \cdot \qquad \cdot \qquad \cdot \end{array}$ |
| MA(r;2,0): Second order backward         | $\begin{array}{c} x-2 \qquad x-1 \qquad x \\ \hline \cdot \qquad \cdot \qquad \cdot \end{array}$ |
| MA(r;0,2): Second order forward          | $\begin{array}{c} x \qquad x+1 \qquad x+2 \\ \hline \cdot \qquad \cdot \qquad \cdot \end{array}$ |

As indicated previously, we will restrict our attention to the cases  $r=1,2$ ; this results in a set of ten models of which we will analyze a subset.

2.1 The model MA(1;1,0). From equation (2.1) we obtain

$$\begin{aligned}\tilde{z}_{x,t} &= (1 + \sum_{n_1=-1}^0 \sum_{k=1}^1 \psi_{n_1,k} F_{x,t}^{n_1 B_k}) a_{x,t} \\ &= a_{x,t} + \psi_{01} a_{x,t-1} + \psi_{-11} a_{x-1,t-1}.\end{aligned}\quad (2.1.1)$$

For convenience, and to indicate that we are using a finite set of weight parameters, we change symbols, letting  $\psi_{01} = -\theta_1$  and  $\psi_{-11} = -\theta_2$ . Then (2.1.1) becomes

$$\tilde{z}_{x,t} = (1 - \theta_1 B_t - \theta_2 B_x B_t) a_{x,t} = \Theta(B_x, B_t) a_{x,t}. \quad (2.1.2)$$

Multiplying through (2.1.2) by  $\tilde{z}_{x-l,t-k}$  we get

$$\tilde{z}_{x-l,t-k} \tilde{z}_{x,t} = \tilde{z}_{x-l,t-k} a_{x,t} - \theta_1 \tilde{z}_{x-l,t-k} a_{x,t-1} - \theta_2 \tilde{z}_{x-l,t-k} a_{x-1,t-1} \quad (2.1.3)$$

On taking expected values in (2.1.3) we find the variance of the process is

$$\gamma_{00} = \sigma_z^2 = (1 + \theta_1^2 + \theta_2^2) \sigma_a^2 \quad (2.1.4)$$

and

$$\gamma_{01} = -\theta_1 \sigma_a^2, \quad \gamma_{10} = \theta_1 \theta_2 \sigma_a^2, \quad \gamma_{11} = -\theta_2 \sigma_a^2,$$

while all other autocovariances are zero. Thus the autocorrelation function is

$$\rho_{01} = \frac{-\theta_1}{1 + \theta_1^2 + \theta_2^2}, \quad \rho_{10} = \frac{\theta_1 \theta_2}{1 + \theta_1^2 + \theta_2^2}, \quad \rho_{11} = \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2}, \quad (2.1.5)$$

all other being zero.

To illustrate the technique of obtaining the autocovariances using the autocovariance generating function (1.9),

$$\Psi(B_x B_t) = \Theta(B_x, B_t) = 1 - \theta_1 B_t - \theta_2 B_x B_t \quad (2.1.6)$$

is substituted into (1.10) yielding

$$\begin{aligned} \gamma(B_x, B_t) &= \sigma_a^2 (1 - \theta_1 B_t - \theta_2 B_x B_t) (1 - \theta_1 B_t^{-1} - \theta_2 B_x^{-1} B_t^{-1}) \quad (2.1.7) \\ &= \sigma_a^2 [-\theta_2 B_x^{-1} B_t^{-1} + \theta_1 \theta_2 B_x^{-1} - \theta_1 B_t^{-1} + (1 + \theta_1^2 + \theta_2^2) \\ &\quad - \theta_1 B_t + \theta_1 \theta_2 B_x - \theta_2 B_x B_t]. \end{aligned}$$

Comparing (2.1.7) with (1.9) and noting that  $\gamma_{-1-1} = \gamma_{11}$ ,  $\gamma_{-10} = \gamma_{10}$ , and  $\gamma_{0-1} = \gamma_{01}$ , we obtain the same results as listed in (2.1.4).

Substituting  $B_x = e^{-i2\pi g}$  and  $B_t = e^{-i2\pi f}$  into the autocovariance generating function (1.9), with  $\Psi(B_x, B_t)$  as in (2.1.6), we obtain one half of the power spectrum. Thus the spectrum is

$$\begin{aligned} p(g, f) &= 2\sigma_a^2 |\Psi(e^{-i2\pi g}, e^{-i2\pi f})|^2 \quad (2.1.8) \\ &= 2\sigma_a^2 |1 - \theta_1 e^{-i2\pi f} - \theta_2 e^{-i2\pi g} e^{-i2\pi f}|^2 \\ &= 2\sigma_a^2 [1 + \theta_1^2 + \theta_2^2 + 2\theta_1 \theta_2 \cos 2\pi g \\ &\quad - 2\{\theta_1 \cos 2\pi f + \theta_2 \cos 2\pi(f+g)\}] \\ &\quad 0 \leq f \leq \frac{1}{2}; \quad 0 \leq g \leq \frac{1}{2} \end{aligned}$$

For invertibility, the generating function  $\Pi(B_x, B_t) = \Theta^{-1}(B_x, B_t)$  must converge for  $|B_t| \leq 1$  and  $|B_x| \leq 1$ . Since

$$\Pi(B_x, B_t) = [1 - (\theta_1 + \theta_2 B_x) B_t]^{-1} = \sum_{j=0}^{\infty} (\theta_1 + \theta_2 B_x)^j B_t^j,$$

we see that for invertibility,

$$|\theta_1 + \theta_2 B_x| < 1 \text{ and } |B_x| \leq 1.$$

Consequently, the parameters of the MA(1;1,0) process must satisfy  $|\theta_1| + |\theta_2| < 1$  to ensure invertibility. If the autocorrelations are known we can solve for the parameters  $\theta_1$  and  $\theta_2$  from equation (2.1.5) even though they are nonlinear; for other models we may

need to use an iterative process. In the present case,

$$\theta_1 = \frac{-\rho_{10}}{\rho_{11}}, \quad \theta_2 = \frac{-\rho_{10}}{\rho_{01}} \quad (2.1.9)$$

If any of the autocorrelations are zero, then from (2.1.5),  $\theta_1$  or  $\theta_2$  is zero, the model reduces to the zero dimensional case and we need to solve a quadratic equation for the nonzero parameters.

Finally, we wish to point out that the analysis of the forward model MA(1;0,1) given by

$$\begin{aligned} \tilde{z}_{x,t} &= (1 + \sum_{n_1=0}^1 \sum_{k=1}^1 \psi_{n_1,k} F_x^{n_1} B_t^k) a_{x,t} \quad (2.1.10) \\ &= a_{x,t} + \psi_{01} a_{x,t-1} + \psi_{11} a_{x+1,t-1} \\ &= (1 - \theta_1 B_t - \theta_3 F_x B_t) a_{x,t} \\ &\quad \psi_{01} = -\theta_1; \quad \psi_{11} = -\theta_3 \end{aligned}$$

is very similar to that of the MA(1;1,0) model. The results are analogous with the parameter  $\theta_2$  replaced by  $\theta_3$  and the operator  $B_x$  replaced by  $F_x$ .

2.2 The model MA(2;1,0). From (1.6) we obtain

$$\begin{aligned} \tilde{z}_{x,t} &= (1 + \sum_{n_1=-1}^0 \sum_{k=1}^2 \psi_{n_1,k} F_x^{n_1} B_t^k) a_{x,t} \quad (2.2.1) \\ &= a_{x,t} + \psi_{01} a_{x,t-1} + \psi_{-11} a_{x-1,t-1} + \psi_{-12} a_{x-1,t-2} + \psi_{02} a_{x,t-2} \\ &= (1 - \theta_1 B_t - \theta_2 B_x B_t - \theta_3 B_x B_t^2 - \theta_4 B_t^2) a_{x,t} \\ &= O(B_x, B_t) a_{x,t}, \end{aligned}$$

where  $\psi_{01} = -\theta_1$ ,  $\psi_{-11} = -\theta_2$ ,  $\psi_{-12} = -\theta_3$ , and  $\psi_{02} = -\theta_4$ .



Substituting into the autocovariance generating function results in

$$\gamma(B_x, B_t) = \sigma_a^2 (1 - \theta_1 B_t - \theta_2 B_x B_t - \theta_3 B_x B_t^2 - \theta_4 B_t^2) \\ (1 - \theta_1 B_t^{-1} - \theta_2 B_x^{-1} - \theta_3 B_x^{-1} B_t^{-2} - \theta_4 B_t^{-2}).$$

Multiplying out this expression and comparing the result with (1.9) we find the variance of the process is

$$\gamma_{00} = \sigma_z^2 = (1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2) \sigma_a^2 \quad (2.2.2)$$

and

$$\gamma_{01} = (-\theta_1 + \theta_2 \theta_3 + \theta_1 \theta_4) \sigma_a^2, \quad \gamma_{10} = (\theta_1 \theta_2 + \theta_3 \theta_4) \sigma_a^2, \\ \gamma_{11} = (-\theta_2 + \theta_1 \theta_3) \sigma_a^2, \quad \gamma_{-11} = \theta_2 \theta_4 \sigma_a^2, \\ \gamma_{02} = -\theta_4 \sigma_a^2, \quad \gamma_{12} = -\theta_3 \sigma_a^2$$

while all other autocovariances are zero. Thus the autocorrelation function is

$$\rho_{01} = \frac{-\theta_1 + \theta_2 \theta_3 + \theta_1 \theta_4}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2}, \quad \rho_{10} = \frac{\theta_1 \theta_2 + \theta_3 \theta_4}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2}, \\ \rho_{11} = \frac{-\theta_2 + \theta_1 \theta_3}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2}, \quad \rho_{-11} = \frac{\theta_2 \theta_4}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2}, \\ \rho_{02} = \frac{-\theta_4}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2}, \quad \rho_{12} = \frac{-\theta_3}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2}. \quad (2.2.3)$$

From (1.11), the power spectrum for this model is given by

$$p(g, f) = 2\sigma_a^2 |1 - \theta_1 e^{-i2\pi f} - \theta_2 e^{-i2\pi g} - \theta_3 e^{-i2\pi f} - \theta_4 e^{-i4\pi f}|^2 \\ 0 < f < \frac{1}{2}; \quad 0 < g < \frac{1}{2} \quad (2.2.4)$$

For invertibility,  $\Pi(B_x, B_t) = \Theta^{-1}(B_x, B_t)$  must converge for  $|B_t| \leq 1$  and  $|B_x| \leq 1$ . Equivalently, the roots of

$$\Theta(B_x, B_t) = 1 - (\theta_1 + \theta_2 B_x) B_t - (\theta_4 + \theta_3 B_x) B_t^2$$

must be outside the region  $|B_t| \leq 1$  and  $|B_x| \leq 1$ . Hence the parameters must satisfy:

$$\begin{aligned} \theta_1 + \theta_2 + \theta_3 + \theta_4 &\leq 1, \quad -\theta_1 - \theta_2 + \theta_3 + \theta_4 \leq 1, \\ \theta_1 - \theta_2 - \theta_3 + \theta_4 &\leq 1, \quad -\theta_1 + \theta_2 - \theta_3 + \theta_4 \leq 1, \\ |\theta_3| + |\theta_4| &\leq 1. \end{aligned} \quad (2.2.5)$$

Note that as a consequence of these inequalities it follows that  $|\theta_1| + |\theta_2| \leq 2$ .

Setting  $\theta_3 = \theta_4 = 0$ , we obtain the model MA(1;1,0) discussed previously. From (2.2.5) with  $\theta_3 = \theta_4 = 0$  we obtain  $|\theta_1| + |\theta_2| \leq 1$ , which agrees with the invertibility condition for the MA(1;1,0) model. In addition, note that with  $\theta_2 = \theta_3 = \theta_4 = 0$  we obtain the MA(1;0,0) ~MA(1) process of Box and Jenkins, and (2.2.5) implies

$$\theta_1 + \theta_4 \leq 1, \quad \theta_4 - \theta_1 \leq 1, \quad |\theta_4| \leq 1,$$

which is also in agreement. If in (2.2.3) the autocorrelations are known, we get a nonlinear system of six equations to solve for the parameter  $\theta_i$ ,  $1 \leq i \leq 4$ . Certain dependencies exist among the autocorrelations as in the previous model, and the resultant system may be solved iteratively.

2.3 The model MA(1;2,0). From (1.6) we obtain

$$\begin{aligned} \tilde{z}_{x,t} &= (1 + \sum_{n_1=-2}^0 \sum_{k=1}^1 \psi_{n_1,k} F_{x,t}^{n_1 k}) a_{x,t} \\ &= a_{x,t} + \psi_{01} a_{x,t-1} + \psi_{-11} a_{x-1,t-1} + \psi_{-21} a_{x-2,t-1} \\ &= a_{x,t} - \theta_1 a_{x,t-1} - \theta_2 a_{x-1,t-1} - \theta_3 a_{x-2,t-1} \end{aligned} \quad (2.3.1)$$

$$=(1-\theta_1 B_t - \theta_2 B_x B_t - \theta_3 B_x^2 B_t) a_{x,t}$$

$$=O(B_x, B_t) a_{x,t},$$

where  $\psi_{01} = -\theta_1$ ,  $\psi_{11} = -\theta_2$ , and  $\psi_{21} = -\theta_3$ . Substituting into the autocovariance generating function we find

$$\begin{aligned} \gamma(B_x, B_t) &= \sigma_a^2 (1 - \theta_1 B_t - \theta_2 B_x B_t - \theta_3 B_x^2 B_t) \\ &\quad (1 - \theta_1 B_t^{-1} - \theta_2 B_x^{-1} B_t^{-1} - \theta_3 B_x^{-2} B_t^{-1}). \end{aligned}$$

Multiplying out this expression and comparing the result with (1.9), we find the variance of the process is

$$\gamma_{00} = \sigma_z^2 = (1 + \theta_1^2 + \theta_2^2 + \theta_3^2) \sigma_a^2$$

and

$$\gamma_{01} = -\theta_1 \sigma_a^2, \quad \gamma_{11} = -\theta_2 \sigma_a^2$$

$$\gamma_{10} = (\theta_1 \theta_2 + \theta_2 \theta_3) \sigma_a^2, \quad \gamma_{21} = -\theta_3 \sigma_a^2$$

$$\gamma_{20} = \theta_1 \theta_3 \sigma_a^2,$$

while all other autocovariances are zero. Thus the autocorrelations are given by

$$\begin{aligned} \rho_{01} &= \frac{-\theta_1}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2}, \quad \rho_{11} = \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2}, \\ \rho_{10} &= \frac{\theta_1 \theta_2 + \theta_2 \theta_3}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2}, \quad \rho_{21} = \frac{-\theta_3}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2}, \end{aligned}$$

$$\rho_{20} = \frac{\theta_1 \theta_3}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2}$$



From (1.11), the power spectrum for this model is given by

$$p(g, f) = 2\sigma_a^2 |1 - \theta_1 e^{-i2\pi f} - \theta_2 e^{-i2\pi g} - \theta_3 e^{-i2\pi f} - \theta_3 e^{-i4\pi g} - i2\pi f}|^2 \quad (2.3.4)$$

$$0 \leq f \leq \frac{1}{2}; \quad 0 \leq g \leq \frac{1}{2}.$$

For invertibility,  $\Pi(B_x, B_t) = \Theta^{-1}(B_x, B_t)$  must converge for  $|B_t| \leq 1$  and  $|B_x| \leq 1$ . Since  $\Theta(B_x, B_t) = 1 - (\theta_1 + \theta_2 B_x + \theta_3 B_x^2) B_t$ , invertibility is ensured if  $|\theta_1 + \theta_2 B_x + \theta_3 B_x^2| < 1$  and  $|B_x| \leq 1$ . Thus the parameters must satisfy

$$|\theta_1 + \theta_2 + \theta_3| < 1, \quad |\theta_1 - \theta_2 + \theta_3| < 1,$$

$$\text{and if } \frac{\theta_2}{2\theta_3} < 1, \text{ then } \left| \theta_1 - \frac{\theta_2^2}{4\theta_3} \right| < 1.$$

2.4 The model MA(1;1,1). From 1.6 we obtain

$$\tilde{z}_{x,t} = (1 + \sum_{n_1=-1}^1 \sum_{k=1}^1 \psi_{n_1, k} F_{x,t}^{n_1 k} B_t^k) a_{x,t} \quad (2.4.1)$$

$$= a_{x,t} + \psi_{01} a_{x,t-1} + \psi_{-11} a_{x-1,t-1} + \psi_{11} a_{x+1,t-1}$$

$$= a_{x,t} - \theta_1 a_{x,t-1} - \theta_2 a_{x-1,t-1} - \theta_3 a_{x+1,t-1}$$

$$= (1 - \theta_1 B_t - \theta_2 B_x B_t - \theta_3 B_x^{-1} B_t) a_{x,t}$$

$$= \Theta(B_x, B_t) a_{x,t},$$

where  $\psi_{01} = -\theta_1$ ,  $\psi_{-11} = -\theta_2$ , and  $\psi_{11} = -\theta_3$ . Substituting into the autocovariance generating function

$$\gamma(B_x, B_t) = \sigma_a^2 (1 - \theta_1 B_t - \theta_2 B_x B_t - \theta_3 B_x^{-1} B_t) (1 - \theta_1 B_t^{-1} - \theta_2 B_x^{-1} B_t^{-1} - \theta_3 B_x B_t^{-1}),$$

and comparing with (1.9), we find the variance of the process is

$$\gamma_{00} = \sigma_z^2 = (1 + \theta_1^2 + \theta_2^2 + \theta_3^2) \sigma_a^2$$

and

$$\begin{aligned}
\gamma_{01} &= -\theta_1 \sigma_a^2, & \gamma_{11} &= -\theta_2 \sigma_a^2, \\
\gamma_{10} &= (\theta_1 \theta_2 + \theta_1 \theta_3) \sigma_a^2, & \gamma_{-11} &= -\theta_3 \sigma_a^2, \\
\gamma_{20} &= \theta_2 \theta_3 \sigma_a^2,
\end{aligned} \tag{2.4.2}$$

while all other autocovariances are zero. Thus the autocorrelation function is

$$\begin{aligned}
\rho_{01} &= \frac{-\theta_1}{1+\theta_1^2+\theta_2^2+\theta_3^2}, & \rho_{11} &= \frac{-\theta_2}{1+\theta_1^2+\theta_2^2+\theta_3^2} \\
\rho_{10} &= \frac{\theta_1(\theta_2+\theta_3)}{1+\theta_1^2+\theta_2^2+\theta_3^2}, & \rho_{-11} &= \frac{-\theta_3}{1+\theta_1^2+\theta_2^2+\theta_3^2}, \\
\rho_{20} &= \frac{\theta_2 \theta_3}{1+\theta_1^2+\theta_2^2+\theta_3^2}.
\end{aligned} \tag{2.4.3}$$

From (1.11), the power spectrum is given by

$$p(g, f) = 2\sigma_a^2 |1 - \theta_1 e^{-i2\pi f} - \theta_2 e^{-i2\pi g} - \theta_3 e^{i2\pi g} e^{-i2\pi f}|^2, \tag{2.4.4}$$

$$0 \leq f < \frac{1}{2}; \quad 0 \leq g \leq \frac{1}{2}.$$

For invertibility,  $\Pi(B_x, B_t) = \theta^{-1}(B_x, B_t)$  must converge on  $\sum_{i=-1}^1 S_i$ ;

since

$$\theta(B_x, B_t) = 1 - (\theta_1 + \theta_2 B_x + \theta_3 F_x) B_t, \text{ we obtain}$$

$$|\theta_1 + \theta_2 + \theta_3| < 1, \quad |\theta_1 + \theta_2 - \theta_3| < 1,$$

$$|\theta_1 - \theta_2 + \theta_3| < 1, \quad |\theta_1 - \theta_2 - \theta_3| < 1,$$

or, equivalently,  $|\theta_1| + |\theta_2| + |\theta_3| < 1$  as conditions on the parameters.

3. Two-Dimensional Time Series. In this section we consider MA models of two-dimensional time series with temporal and spatial orders not exceeding two. For convenience, we let  $x = x_1$ ,  $y = y_1$  and

denote by  $MA(r;p,q)$ , where  $p=(p_1,p_2)$ ,  $q=(q_1,q_2)$ , the MA process of temporal order  $r$  and spatial order  $p_1+q_1$  in the  $x$ -direction and  $p_2+q_2$  in the  $y$ -direction.

From (1.1) the general model has the form

$$\tilde{z}_{x,y,t} = \sum_{n_2=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} \sum_{k=1}^{\infty} \psi_{n_1,n_2,k} a_{x+n_1,y+n_2,t-k} + a_{x,y,t}. \quad (3.1)$$

The special case of (3.1) in which only a finite number of the coefficients are nonzero results in

$$\tilde{z}_{x,y,t} = \sum_{n_2=-p_2}^{q_2} \sum_{n_1=-p_1}^{q_1} \sum_{k=1}^r \psi_{n_1,n_2,k} a_{x+n_1,y+n_2,t-k} + a_{x,y,t}, \quad (3.2)$$

or in terms of shift operators,

$$\begin{aligned} \tilde{z}_{x,y,t} &= (1 + \sum_{n_2=-p_2}^{q_2} \sum_{n_1=-p_1}^{q_1} \sum_{k=1}^r \psi_{n_1,n_2,k} F_x^{n_1} F_y^{n_2} B_t^k) a_{x,y,t} \\ &= \Psi(B_x, B_y, B_t) a_{x,y,t}. \end{aligned} \quad (3.3)$$

In addition to the zero-dimensional model corresponding to  $p=q=0$ , and the ten one-dimensional models in each direction resulting by letting  $p_1=q_1=0$  or  $p_2=q_2=0$ , there are fifty general "spatial" models described by (3.3) where  $1 \leq p_i+q_i \leq 2$ ,  $i=1,2$ , and  $1 \leq r \leq 2$ . In what follows, we examine three such models.

3.1 The model  $MA(1;1,1,0,0)$ . From (3.3) we obtain

$$\begin{aligned} \tilde{z}_{x,y,t} &= (1 + \sum_{n_2=-1}^0 \sum_{n_1=-1}^0 \sum_{k=1}^1 \psi_{n_1,n_2,k} F_x^{n_1} F_y^{n_2} B_t^k) a_{x,y,t} \quad (3.1.1) \\ &= a_{x,y,t} + \psi_{001} a_{x,y,t-1} + \psi_{-101} a_{x-1,y,t-1} \\ &\quad + \psi_{0-11} a_{x,y-1,t-1} + \psi_{-1-11} a_{x-1,y-1,t-1} \\ &= a_{x,y,t} - \theta_1 a_{x,y,t-1} - \theta_2 a_{x-1,y,t-1} - \theta_3 a_{x,y-1,t-1} \\ &\quad - \theta_4 a_{x-1,y-1,t-1} \\ &= (1 - \theta_1 B_t - \theta_2 B_x B_t - \theta_3 B_y B_t - \theta_4 B_x B_y B_t) a_{x,y,t} \end{aligned}$$

$$= \Theta(B_x, B_y, B_t) a_{x,y,t},$$

where  $\psi_{001} = -\theta_1$ ,  $\psi_{-101} = -\theta_2$ ,  $\psi_{0-11} = -\theta_3$ , and  $\psi_{-1-11} = -\theta_4$ .

Substituting into the autocovariance generating function

$$\begin{aligned} \gamma(B_x, B_y, B_t) = & \sigma_a^2 (1 - \theta_1 B_t - \theta_2 B_x B_t - \theta_3 B_y B_t - \theta_4 B_x B_y B_t) \times \\ & (1 - \theta_1 B_t^{-1} - \theta_2 B_t^{-1} - \theta_3 B_t^{-1} B_t^{-1} - \theta_4 B_t^{-1} B_t^{-1} B_t^{-1}) \end{aligned}$$

and comparing with (1.9), we find the variance of the process is

$$\gamma_{000} = \sigma_z^2 = (1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2) \sigma_a^2$$

and

$$\begin{aligned} \gamma_{001} &= -\theta_1 \sigma_a^2, \quad \gamma_{100} = (\theta_1 \theta_2 + \theta_3 \theta_4) \sigma_a^2, \\ \gamma_{101} &= -\theta_2 \sigma_a^2, \quad \gamma_{010} = (\theta_1 \theta_3 + \theta_2 \theta_4) \sigma_a^2, \\ \gamma_{011} &= -\theta_3 \sigma_a^2, \quad \gamma_{110} = \theta_1 \theta_4 \sigma_a^2, \\ \gamma_{111} &= -\theta_4 \sigma_a^2, \quad \gamma_{1-10} = \theta_2 \theta_3 \sigma_a^2, \end{aligned} \quad (3.1.2)$$

while all other autocovariances are zero. Thus the autocorrelation function is given by

$$\begin{aligned} \rho_{001} &= \frac{-\theta_1}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2}, \quad \rho_{100} = \frac{\theta_1 \theta_2 + \theta_3 \theta_4}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2}, \\ \rho_{101} &= \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2}, \quad \rho_{010} = \frac{\theta_1 \theta_3 + \theta_2 \theta_4}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2}, \\ \rho_{011} &= \frac{-\theta_3}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2}, \quad \rho_{110} = \frac{\theta_1 \theta_4}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2}, \\ \rho_{111} &= \frac{-\theta_4}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2}, \quad \rho_{1-10} = \frac{\theta_2 \theta_3}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2}. \end{aligned} \quad (3.1.3)$$

On substituting  $B_x = e^{-i2\pi g_1}$ ,  $B_y = e^{-i2\pi g_2}$ , and  $B_t = e^{-i2\pi f}$  into (1.11), we find the power spectrum of the process is given by

$$p(g, f) = 2\sigma_a^2 |1 - (\theta_1 - \theta_2 e^{-i2\pi g_1} - \theta_3 e^{-i2\pi g_2} - \theta_4 e^{-i2\pi g_1 - i2\pi g_2}) e^{-i2\pi f}|^2. \quad (3.1.4)$$



For invertibility,

$$\begin{aligned} \|(B_x, B_y, B_t) &= \theta^{-1} (B_x, B_y, B_t) \\ &= [1 - (\theta_1 + \theta_2 B_x + \theta_3 B_y + \theta_4 B_x B_y) B_t]^{-1} \end{aligned}$$

must converge on  $\sum_{i=-2}^0 S_i$ . Hence, the parameters must satisfy

$$\begin{aligned} |\theta_1 + \theta_2 + \theta_3 + \theta_4| < 1, \quad |\theta_1 + \theta_2 - \theta_3 - \theta_4| < 1, \\ |\theta_1 - \theta_2 - \theta_3 + \theta_4| < 1, \quad |\theta_1 - \theta_2 + \theta_3 - \theta_4| < 1, \end{aligned}$$

to ensure invertibility.

3.2 The model MA(1;1,1,1,1). From (3.3) we find

$$\begin{aligned} \tilde{z}_{x,y,t} &= (1 + \sum_{n_2=-1}^1 \sum_{n_1=-1}^1 \sum_{k=1}^1 \psi_{n_1, n_2, k}^F a_{x,y,t}^{n_1 n_2 k}) a_{x,y,t} \\ &= a_{x,y,t} + \psi_{-1-11} a_{x-1,y-1,t-1} + \psi_{0-11} a_{x,y-1,t-1} \\ &\quad + \psi_{1-11} a_{x+1,y-1,t-1} + \psi_{-101} a_{x-1,y,t-1} \\ &\quad + \psi_{001} a_{x,y,t} + \psi_{101} a_{x+1,y,t-1} + \psi_{-111} a_{x-1,y+1,t-1} \\ &\quad + \psi_{011} a_{x,y+1,t-1} + \psi_{111} a_{x+1,y+1,t-1}. \end{aligned} \quad (3.2.1)$$

Consider the special case where  $\psi_{-1-11} = \psi_{1-11} = \psi_{-111} = \psi_{111} = 0$ .  
Letting  $\psi_{001} = -\theta_1$ ,  $\psi_{-101} = -\theta_2$ ,  $\psi_{101} = -\theta_3$ ,  $\psi_{0-11} = -\theta_4$  and  $\psi_{011} = -\theta_5$ ,  
we obtain

$$\begin{aligned} \tilde{z}_{x,y,t} &= a_{x,y,t} - \theta_1 a_{x,y,t-1} - \theta_2 a_{x-1,y,t-1} - \theta_3 a_{x+1,y,t-1} \\ &\quad - \theta_4 a_{x,y-1,t-1} - \theta_5 a_{x,y+1,t-1} \\ &= [1 - (\theta_1 + \theta_2 B_x + \theta_3 B_x^{-1} + \theta_4 B_y + \theta_5 B_y^{-1}) B_t] a_{x,y,t} \\ &= \theta (B_x, B_y, B_t) a_{x,y,t}. \end{aligned}$$

Substituting into the autocovariance function

$$\gamma(B_x, B_y, B_t) = \sigma_a^2 (1 - \theta_1 B_t - \theta_2 B_x B_t - \theta_3 B_x^{-1} B_t - \theta_4 B_y B_t - \theta_5 B_y^{-1} B_t) \times \\ (1 - \theta_1 B_t^{-1} - \theta_2 B_x^{-1} B_t^{-1} - \theta_3 B_x B_t^{-1} - \theta_4 B_y^{-1} B_t^{-1} - \theta_5 B_y B_t^{-1})$$

and comparing with (1.9), we find the variance of the process is

$$\gamma_{000} = \sigma_z^2 = (1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 + \theta_5^2) \sigma_a^2$$

and

$$\begin{aligned} \gamma_{001} &= -\theta_1 \sigma_a^2 & \gamma_{100} &= \theta_1 (\theta_2 + \theta_3) \sigma_a^2 \\ \gamma_{101} &= -\theta_2 \sigma_a^2 & \gamma_{010} &= \theta_1 (\theta_4 + \theta_5) \sigma_a^2 \\ \gamma_{-101} &= -\theta_3 \sigma_a^2 & \gamma_{200} &= \theta_2 \theta_3 \sigma_a^2 \\ \gamma_{011} &= -\theta_4 \sigma_a^2 & \gamma_{020} &= \theta_4 \theta_5 \sigma_a^2 \\ \gamma_{0-11} &= -\theta_5 \sigma_a^2 & \gamma_{110} &= (\theta_2 \theta_5 + \theta_3 \theta_4) \sigma_a^2 \\ \gamma_{1-10} &= (\theta_2 \theta_4 + \theta_3 \theta_5) \sigma_a^2, \end{aligned}$$

while all other autocovariances are zero. From these, we can obtain the autocorrelations.

From (1.11), the power spectrum is given by

$$p(g, f) = |1 - (\theta_1 + \theta_2 e^{-i2\pi g_1} + \theta_3 e^{i2\pi g_1} + \theta_4 e^{-i2\pi g_2} + \theta_5 e^{i2\pi g_2}) e^{-i2\pi f}|^2 \quad (3.2.3)$$

For invertibility,

$$\Pi(B_x, B_y, B_t) = \Theta^{-1}(B_x, B_y, B_t) \\ = [1 - (\theta_1 + \theta_2 B_x + \theta_3 F_x + \theta_4 B_y + \theta_5 F_y) B_t]^2$$

must converge on  $\sum_{i=-2}^2 S_i$ ; hence we need

$$|\theta_1 + \theta_2 B_x + \theta_3 F_x + \theta_4 B_y + \theta_5 F_y| < 1$$

and

$$|B_x| \leq 1, |F_x| \leq 1, |B_y| \leq 1, |F_y| \leq 1.$$

Substituting in the values  $\pm 1$  for the (dummy) variables results in a system of inequalities that must be satisfied by the

parameters to ensure invertibility. Aroian finds the values of  $\theta_i$  in terms of the essential correlations  $\rho_{001}$ ,  $\rho_{10}$ ,  $\rho_{20}$ ,  $\rho_{01}$ , and  $\rho_{02}$ , and  $\sigma_a^2$ .

3.3 The model MA(2;1,1,1,1). We consider only a special case of this model consisting of the model in section 3.2 except with a lag of two in time. We obtain

$$\begin{aligned} z_{x,y,t} = & a_{x,y,t} + \psi_{0-11} a_{x,y-1,t-1} + \psi_{0-12} a_{x,y-1,t-2} \\ & + \psi_{-101} a_{x-1,y,t-1} + \psi_{-102} a_{x-1,y,t-2} \\ & + \psi_{001} a_{x,y,t-1} + \psi_{002} a_{x,y,t-2} + \psi_{101} a_{x+1,y,t-1} \\ & + \psi_{102} a_{x+1,y,t-2} + \psi_{011} a_{x,y+1,t-1} + \psi_{012} a_{x,y+1,t-2}. \end{aligned} \quad (3.3.1)$$

Letting  $\psi_{001} = -\theta_1$ ,  $\psi_{-101} = -\theta_2$ ,  $\psi_{101} = -\theta_3$ ,  $\psi_{0-11} = -\theta_4$ ,  $\psi_{011} = -\theta_5$ ,  $\psi_{002} = -\theta_6$ ,  $\psi_{-102} = -\theta_7$ ,  $\psi_{102} = -\theta_8$ ,  $\psi_{0-12} = -\theta_9$ ,  $\psi_{012} = -\theta_{10}$ , we can rewrite (3.3.1) as

$$\begin{aligned} z_{x,y,t} = & [1 - (\theta_1 + \theta_2 B_x + \theta_3 F_x + \theta_4 B_y + \theta_5 F_y) B_t - \\ & (\theta_6 + \theta_7 B_x + \theta_8 F_x + \theta_9 B_y + \theta_{10} F_y) B_t^2] a_{x,y,t} \\ = & (B_x, B_y, B_t) a_{x,y,t}. \end{aligned} \quad (3.3.2)$$

Substituting into the autocovariance generating function we find the variance of the process is

$$\gamma_{000} = \sigma_z^2 = (1 + \sum_{i=1}^{10} \theta_i^2) \sigma_a^2$$

and



$$\begin{aligned}
\gamma_{001} &= (-\theta_{11} + \theta_{16} + \theta_{27} + \theta_{38} + \theta_{49} + \theta_{510})\sigma_a^2, \\
\gamma_{100} &= (\theta_{12} + \theta_{13} + \theta_{67} + \theta_{68})\sigma_a^2, \quad \gamma_{1-10} = (\theta_{24} + \theta_{35} + \theta_{79} + \theta_{810})\sigma_a^2 \\
\gamma_{010} &= (\theta_{14} + \theta_{15} + \theta_{69} + \theta_{610})\sigma_a^2, \quad \gamma_{111} = (\theta_{39} + \theta_{57})\sigma_a^2, \\
\gamma_{101} &= (-\theta_{11} + \theta_{17} + \theta_{36})\sigma_a^2, \quad \gamma_{-111} = (\theta_{29} + \theta_{58})\sigma_a^2, \\
\gamma_{-101} &= (-\theta_{32} + \theta_{61} + \theta_{8})\sigma_a^2, \quad \gamma_{1-11} = (\theta_{310} + \theta_{47})\sigma_a^2 \\
\gamma_{011} &= (-\theta_{41} + \theta_{95} + \theta_{6})\sigma_a^2, \quad \gamma_{-1-11} = (\theta_{210} + \theta_{48})\sigma_a^2 \\
\gamma_{0-11} &= (-\theta_{51} + \theta_{104} + \theta_{6})\sigma_a^2, \quad \gamma_{012} = -(\theta_9 + \theta_{10})\sigma_a^2 \\
\gamma_{002} &= -\theta_6\sigma_a^2, \quad \gamma_{021} = (\theta_{410} + \theta_{59})\sigma_a^2 \\
\gamma_{200} &= (\theta_{23} + \theta_{78})\sigma_a^2, \quad \gamma_{102} = -(\theta_7 + \theta_8)\sigma_a^2, \\
\gamma_{020} &= (\theta_{45} + \theta_{910})\sigma_a^2, \quad \gamma_{201} = (\theta_{28} + \theta_{37})\sigma_a^2, \\
\gamma_{110} &= (\theta_{25} + \theta_{34} + \theta_{710} + \theta_{89})\sigma_a^2,
\end{aligned} \tag{3.3.3}$$

all other autocovariances being zero; from these we can determine the autocorrelations.

From (1.11) the power spectrum is given by

$$\begin{aligned}
p(g, f) &= 2\sigma_a^2 \left| 1 - (\theta_1 + \theta_2 e^{-i2\pi g_1} + \theta_3 e^{i2\pi g_1} + \theta_4 e^{-i2\pi g_2} + \theta_5 e^{i2\pi g_2}) e^{-2\pi f} \right. \\
&\quad \left. - (\theta_6 + \theta_7 e^{-i2\pi g_1} + \theta_8 e^{i2\pi g_1} + \theta_9 e^{-i2\pi g_2} + \theta_{10} e^{i2\pi g_2}) e^{-4\pi f} \right|^2.
\end{aligned}$$

For invertibility,  $\prod_{i=1}^2 (B_x, B_y, B_t) = \theta^{-1} (B_x, B_y, B_t)$  must converge on

$X_{i=-2} S_i$ . Alternatively, the roots of  $\Theta(B_x, B_y, B_t)$  must be outside

$X_{i=-2} S_i$ , and we find

$$\theta_6 + \theta_7 B_x + \theta_8 F_x + \theta_9 B_y + \theta_{10} F_y + \theta_1 + \theta_2 B_x + \theta_3 F_x + \theta_4 B_y + \theta_5 F_y < 1,$$

$$\theta_6 + \theta_7 B_x + \theta_8 F_x + \theta_9 B_y + \theta_{10} F_y - \theta_1 - \theta_2 B_x - \theta_3 F_x - \theta_4 B_y - \theta_5 F_y < 1,$$

and

$$|\theta_6 + \theta_7 B_x + \theta_8 F_x + \theta_9 B_y + \theta_{10} F_y| < 1.$$

Setting the variables equal to 1 leads to a system of inequalities in the ten parameters.

4. Conclusions. We have defined the properties of a general moving average model of  $m$ -dimensional time series. Specific one-dimensional and two-dimensional models have been investigated. Further work remains to be done on moving average models of  $n$  time series in  $m$ -dimensions.

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